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A Darboux transformation and an exact solution for the relativistic Toda lattice equation

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Abstract
A Darboux transformation for the relativistic Toda lattice hierarchy is constructed. As an application, an exact solution of the relativistic Toda lattice equation is presented.

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1. Introduction

The aim of this paper is to establish a Darboux transformation for the relativistic Toda lattice (RTL) equation

$$\ddot{q}_n = (1 + h\dot{q}_{n-1})(1 + h\dot{q}_n)\frac{\exp(q_{n-1} - q_n)}{1 + h^2\exp(q_{n-1} - q_n)} - (1 + h\dot{q}_n)(1 + h\dot{q}_{n+1})\frac{\exp(q_n - q_{n+1})}{1 + h^2\exp(q_n - q_{n+1})}, \tag{1}$$

where $h = \frac{1}{c}$, c is the speed of light, $q_n = q(n)$ is the coordinate of the nth lattice point, and $\dot{q}_n$ means the differentiation of q_n with respect to time t. The RTL equation was first introduced by Ruijsenaars [1] and has been studied by many authors. Bruschi and Ragnisco constructed its Lax representation, recursion operator and Bäcklund transformation [2–4]. Oevel, Fuchssteiner, Zhang and Ragnisco presented its master symmetry and bi-Hamiltonian structure [5]. Suris found its relation to the discrete time Toda lattice [6, 7]. Ohta, Kajiwara, Matusukidara and Satsuma obtained its Casorati determinant solution [8].

The RTL equation (1) can be written as

$$\begin{cases} \dot{p}_n = \exp(q_{n-1} - q_n + hp_{n-1}) - \exp(q_n - q_{n+1} + hp_n), \\ 1 + h\dot{q}_n = \exp(hp_n)(1 + h^2\exp(q_n - q_{n+1})). \end{cases}$$

Furthermore, in terms of the new variables r_n and s_n defined by

$$s_n = \frac{\exp(hp_{n+1}) - 1}{h}, \qquad r_n = \exp(q_n - q_{n+1} + hp_n),$$

the RTL equation (1) takes the form

$$\dot{r}_n = r_n(s_{n-1} - s_n + hr_{n-1} - hr_{n+1}), \qquad \dot{s}_n = (1 + hs_n)(r_n - r_{n+1}). \tag{2}$$

It is apparent that if $h = 0$ equation (2) becomes the well-known Toda lattice equation [9, 10]. Consequently, the RTL equation is a deformation of the Toda lattice equation [11].

2. New zero-curvature representations of the hierarchy of the RTL equations

Let $f = f(n)$ be a lattice function. Following [12] we specify the shift operator E and the inverse E^{-1} of E as

$$(Ef)(n) = f(n+1), \quad (E^{-1}f(n) = f_{n+1}, \qquad n \in \mathbb{Z},$$

and below we always write

$$f^{(k)} = E^k f, \qquad k \in \mathbb{Z}.$$

Consider the discrete spatial spectral problem

$$E\phi = U\phi, \qquad U = \begin{pmatrix} 0 & 1 \\ (h\lambda - 1)r & \lambda + s \end{pmatrix}, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3}$$

where λ is a spectral parameter and $r = r(n)$, $s = s(n)$ are two potentials.

Following [12], we choose the auxiliary spectral problem

$$\phi_{t_m} = V^{[m]}\phi, \quad V^{[m]} = \sum_{i=0}^{m} \begin{pmatrix} a_i & b_i \\ (h\lambda - 1)c_i & -a_i \end{pmatrix} \lambda^{m-i} + \begin{pmatrix} b_{m+1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad m \geqslant 1, \tag{4}$$

where a_i, b_i, c_i are uniquely determined by the following initial condition and recursion relation:

$$a_0 = -\tfrac{1}{2}, \quad b_0 = 0, \quad c_0 = 0, \tag{5}$$

$$c_{k+1} - rb_{k+1}^{(1)} = 0, \qquad k \geqslant 0, \tag{6}$$

$$b_{k+1}^{(1)} + sb_k^{(1)} + \left(a_k^{(1)} + a_k\right) = 0, \tag{7}$$

$$\left(a_{k+1}^{(1)} - a_{k+1}\right) + s\left(a_k^{(1)} - a_k\right) + h\left(rb_{k+1} - c_{k+1}^{(1)}\right) - \left(rb_k - c_k^{(1)}\right) = 0. \tag{8}$$

In particular, we have

$$\begin{aligned} &a_1 = hr, b_1 = 1, c_1 = r, \\ &a_2 = -h^2r^{(1)}r - h^2r^2 - h^2rr^{(-1)} - hrs - hrs^{(-1)} - r, \\ &b_2 = -hr - hr^{(-1)} - s^{(-1)}, c_2 = -rs - hr^2 - hrr^{(1)}. \end{aligned} \tag{9}$$

It is not difficult to find that the compatible condition of equations (3) and (4),

$$(E\phi)_{t_m} = E\phi_{t_m}, \qquad m \geqslant 1,$$

is equivalent to the discrete zero-curvature equation

$$U_{t_m} = (EV^{[m]})U - UV^{[m]}, \qquad m \geqslant 1, \tag{10}$$

which immediately gives rise to the hierarchy of equations

$$\begin{cases} r_{t_m} = c_{m+1} - rb_{m+1}, \\ s_{t_m} = -c_m^{(1)} + rb_m - s(a_m^{(1)} - a_m), \end{cases} \qquad m \geqslant 1. \tag{11}$$

When $m = 1$ (set $t_1 = t$), the auxiliary spectral problem (4) is

$$\phi_t = V^{[1]}\phi, \qquad V^{[1]} = \begin{pmatrix} -\frac{1}{2}\lambda - hr^{(-1)} - s^{(-1)} & 1 \\ (h\lambda - 1)r & \frac{1}{2}\lambda - hr \end{pmatrix}, \tag{12}$$

and the corresponding equation is nothing but the RTL equation (2). Therefore, (11) is just the hierarchy of the RTL equations.

3. A Darboux transformation for the RTL equation

Recall that a gauge transformation of a spectral problem is called a Darboux transformation if it transforms the spectral problem into another spectral problem of the same type.

Starting from the spectral problem (3), we consider the following Darboux transformation,

$$\tilde{\phi} = T\phi, \tag{13}$$

which transforms (3) into the new spectral problem

$$E\tilde{\phi} = \tilde{U}\tilde{\phi}, \qquad \tilde{U} = T^{(1)}UT^{-1}, \tag{14}$$

where $T = T(n)$ is determined later by requiring that $\tilde{U}$ has the same form as U, replacing r, s with $\tilde{r}, \tilde{s}$ respectively.

T can be determined as follows. Assume T is of the form

$$T = \begin{pmatrix} (1-v)\lambda + u & v \\ (h\lambda - 1)w & \lambda + x \end{pmatrix}, \tag{15}$$

where u, v, w, x are four undetermined functions.

Let $\phi = (\phi_1, \phi_2)^T, \psi = (\psi_1, \psi_2)^T$ be two basic solutions of (3) and λ_1, λ_2 be two solutions of $\det T(\lambda) = 0$. Thus when $\lambda = \lambda_i$ $(i = 1, 2)$ two columns of the matrix

$$T\begin{pmatrix} \phi_1 & \psi_1 \\ \phi_2 & \psi_2 \end{pmatrix} = \begin{pmatrix} (1-v)\lambda\phi_1 + u\phi_1 + v\phi_2 & (1-v)\lambda\psi_1 + u\psi_1 + v\psi_2 \\ (h\lambda - 1)w\phi_1 + (\lambda + x)\phi_2 & (h\lambda - 1)w\psi_1 + (\lambda + x)\psi_2 \end{pmatrix} \tag{16}$$

are linearly dependent. Therefore, without loss of generality we may assume

$$\begin{cases} (1-v)\lambda_i\phi_1(\lambda_i) + u\phi_1(\lambda_i) + v\phi_2(\lambda_i) = \gamma_i[(1-v)\lambda_i\psi_1(\lambda_i) + u\psi_1(\lambda_i) + v\psi_2(\lambda_i)], \\ (h\lambda_i - 1)w\phi_1(\lambda_i) + (\lambda_i + x)\phi_2(\lambda_i) = \gamma_i[(h\lambda_i - 1)w\psi_1(\lambda_i) + (\lambda_i + x)\psi_2(\lambda_i)], \end{cases} \quad i = 1, 2, \tag{17}$$

where γ_1, γ_2 are nonzero constants. Obviously (17) is equivalent to

$$\begin{cases} (1-v)\lambda_i + u + v\chi_i = 0, \\ (h\lambda_i - 1)w + (\lambda_i + x)\chi_i = 0, \end{cases} \tag{18}$$

where

$$\chi_i = \frac{\phi_2(\lambda_i) - \gamma_i\psi_2(\lambda_i)}{\phi_1(\lambda_i) - \gamma_i\psi_1(\lambda_i)}, \qquad i = 1, 2. \tag{19}$$

Solving (18) we obtain

$$\begin{cases} u = \dfrac{\lambda_1\chi_2 - \lambda_2\chi_1}{\chi_1 - \chi_2 - \lambda_1 + \lambda_2}, & v = \dfrac{\lambda_2 - \lambda_1}{\chi_1 - \chi_2 - \lambda_1 + \lambda_2}, \\ w = \dfrac{(\lambda_2 - \lambda_1)\chi_1\chi_2}{\chi_2(h\lambda_1 - 1) - \chi_1(h\lambda_2 - 1)}, & x = \dfrac{\lambda_1\chi_1(h\lambda_2 - 1) - \lambda_2\chi_2(h\lambda_1 - 1)}{\chi_2(h\lambda_1 - 1) - \chi_1(h\lambda_2 - 1)}. \end{cases} \tag{20}$$

Furthermore, from (3) and (19), we get

$$\chi_i^{(1)} = \frac{\phi_2^{(1)}(\lambda_i) - \gamma_i\psi_2^{(1)}(\lambda_i)}{\phi_1^{(1)}(\lambda_i) - \gamma_i\psi_1^{(1)}(\lambda_i)} = \frac{(h\lambda_i - 1)r + (\lambda_i + s)\chi_i}{\chi_i}, \qquad i = 1, 2,$$

which can be written as

$$\chi_i^{(1)} = \frac{\mu_i}{\nu_i}, \qquad i = 1, 2, \tag{21}$$

where

$$\mu_i = (h\lambda_i - 1)r + (\lambda_i + s)\chi_i, \qquad \nu_i = \chi_i, \qquad i = 1, 2. \tag{22}$$

Through a direct calculation we obtain

$$\begin{cases} u^{(1)} = \dfrac{\lambda_1\mu_2\nu_1 - \lambda_2\mu_1\nu_2}{\mu_1\nu_2 - \mu_2\nu_1 - (\lambda_1 - \lambda_2)\nu_1\nu_2}, & v^{(1)} = \dfrac{(\lambda_2 - \lambda_1)\nu_1\nu_2}{\mu_1\nu_2 - \mu_2\nu_1 - (\lambda_1 - \lambda_2)\nu_1\nu_2}, \\ w^{(1)} = \dfrac{(\lambda_2 - \lambda_1)\mu_1\mu_2}{\mu_2\nu_1(h\lambda_1 - 1) - \mu_1\nu_2(h\lambda_2 - 1)}, & x^{(1)} = \dfrac{\lambda_1\mu_1\nu_2(h\lambda_2 - 1) - \lambda_2\mu_2\nu_1(h\lambda_1 - 1)}{\mu_2\nu_1(h\lambda_1 - 1) - \mu_1\nu_2(h\lambda_2 - 1)}, \end{cases} \tag{23}$$

and

$$w = v^{(1)}r, u^{(1)} + sv^{(1)} = x. \tag{24}$$

For T defined by (15) and (20), we have the following proposition.

Proposition 1. *The matrix $\tilde{U}$ defined by $\tilde{U} = T^{(1)}UT^{-1}$ has the same form as U, in which the old potentials r and s are mapped into $\tilde{r}$ and $\tilde{s}$ according to*

$$\tilde{r} = \frac{r - w}{1 - v}, \qquad \tilde{s} = hw^{(1)} + x^{(1)} + s - x - \frac{r - w}{1 - v}hv. \tag{25}$$

Proof. Let us write

$$T^{(1)}UT^* = \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix},$$

where T^* is the adjoint matrix of T and satisfies $T^{-1} = T^*/\det T$.

Through a direct calculation we find that

$$f_{kl}(\lambda_i, n) = 0, \qquad k, l, i = 1, 2. \tag{26}$$

On the other hand, from (20) we know that

$$\det T = ((1 - v)\lambda + u)(\lambda + x) - (h\lambda - 1)vw = (1 - v)(\lambda - \lambda_1)(\lambda - \lambda_2). \tag{27}$$

Therefore, we have

$$T^{(1)}UT^* = (\det T)Q, \tag{28}$$

with

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21}\lambda + Q_1 & Q_{22}\lambda + Q_2 \end{pmatrix}$$

where Q_1, Q_2 and Q_{ij} $(i, j = 1, 2)$ are undetermined functions.

Equation (28) can be written as

$$T^{(1)}U = QT. \tag{29}$$

Substituting

$$T^{(1)} = \begin{pmatrix} (1 - v^{(1)})\lambda + u^{(1)} & v^{(1)} \\ (h\lambda - 1)w^{(1)} & \lambda + x^{(1)} \end{pmatrix} \tag{30}$$

into (29) we arrive at

$$v^{(1)}(h\lambda-1)r = Q_{11}[(1-v\lambda)+u] + Q_{12}(h\lambda-1)w, \tag{31}$$

$$(1-v^{(1)})\lambda + u^{(1)} + (\lambda+s)v^{(1)} = Q_{11}v + Q_{12}(\lambda+x), \tag{32}$$

$$(\lambda+x^{(1)})(h\lambda-1)r = (Q_{21}\lambda+Q_1)[(1-v)\lambda+u] + (Q_{22}\lambda+Q_2)(h\lambda-1)w, \tag{33}$$

$$(h\lambda-1)w^{(1)} + (\lambda+x^{(1)})(\lambda+s) = (Q_{21}\lambda+Q_1)v + (Q_{22}\lambda+Q_2)(\lambda+x). \tag{34}$$

Comparing the coefficients of λ^j $(j=0,1,2)$ gives rise to

$$Q_{11}=0, \qquad Q_{12}=1, \qquad Q_{22}=1, \qquad Q_{21}=h\frac{r-w}{1-v},$$
$$Q_2 = hw^{(1)} + x^{(1)} + s - x - h\frac{r-w}{1-v}v = \tilde{s}, \qquad Q_1 = \frac{w-r}{1-v}.$$

Therefore

$$Q_{21}\lambda + Q_1 = (h\lambda-1)\frac{r-w}{1-v} = (h\lambda-1)\tilde{r}.$$

Hence we obtain

$$Q = \begin{pmatrix} 0 & 1 \\ (h\lambda-1)\tilde{r} & \lambda+\tilde{s} \end{pmatrix} = \tilde{U}.$$

The proof is completed. □

Now let us consider the action of (13) to the spectral problem (12). Under the transformation (13), the spectral problem (11) is transformed to the following new spectral problem:

$$\tilde{\phi}_t = \tilde{V}^{[1]}\tilde{\phi}, \qquad \tilde{V}^{[1]} = (T_t + TV^{[1]})T^{-1}. \tag{35}$$

We claim that if $\phi=(\phi_1,\phi_2)^T$, $\psi=(\psi_1,\psi_2)^T$ are two basic solutions of both (3) and (12), then $\tilde{V}^{[1]}$ has the same form as $V^{[1]}$; that is, the following proposition holds true.

Proposition 2. *Under transformation (13), the corresponding matrix $\tilde{V}^{[1]}$ is of the form*

$$\tilde{V}^{[1]} = \begin{pmatrix} -\frac{1}{2}\lambda - h\tilde{r}^{(-1)} - \tilde{s}^{(-1)} & 1 \\ (h\lambda-1)\tilde{r} & \frac{1}{2}\lambda - h\tilde{r} \end{pmatrix},$$

where $\tilde{r}$ and $\tilde{s}$ are defined by (25).

Proof. If we write

$$(T_t + TV^{[1]})T^* = \begin{pmatrix} g_{11}(\lambda,n) & g_{12}(\lambda,n) \\ g_{21}(\lambda,n) & g_{22}(\lambda,n) \end{pmatrix},$$

then through a straightforward and tedious calculation using the identity

$$\chi_{i,t} = \left(\frac{\phi_2-\gamma_i\psi_2}{\phi_1-\gamma_i\psi_1}\right)_t = (h\lambda_i-1)r + (\lambda_i - hr + hr^{(-1)} + s^{(-1)})\chi_i - \chi_i^2, \tag{36}$$

we know that $g_{11}(\lambda,n)$, $g_{12}(\lambda,n)$, $g_{21}(\lambda,n)$, $g_{22}(\lambda,n)$ are cubic polynomials of λ and

$$g_{kl}(\lambda_i,n)=0, \qquad k,l,i=1,2.$$

Therefore we may assume that

$$(T_t + TV^{[1]})T^* = (\det T)R, \qquad R = \begin{pmatrix} R_{11}\lambda + R_1 & R_{12} \\ R_{21}\lambda + R_2 & R_{22}\lambda + R_3 \end{pmatrix}, \tag{37}$$

where R_1, R_2, R_3 and R_{ij} $(i,j=1,2)$ are undetermined functions.

Equating the coefficients of λ_j $(j = 0, 1, 2)$ on both sides of the equation

$$T_t + TV^{[1]} = RT,$$

we get

$$R_{11} = -\frac{1}{2}, \qquad R_{12} = 1, \qquad R_{22} = \frac{1}{2}, \qquad R_{21} = h\frac{r-w}{1-v} = h\tilde{r},$$

$$R_3 = -h\frac{r-w}{1-v} = -h\tilde{r}, \qquad R_2 = -\frac{r-w}{1-v} = -\tilde{r},$$

$$R_1 = u - hr^{(-1)} + hr^{(-1)}v - s^{(-1)} + s^{(-1)}v - hw - x$$
$$= -h\tilde{r}^{(-1)} - \tilde{s}^{(-1)}.$$

Hence

$$R = \begin{pmatrix} -\frac{1}{2}\lambda - h\tilde{r}^{(-1)} - \tilde{s}^{(-1)} & 1 \\ (h\lambda - 1)\tilde{r} & \frac{1}{2}\lambda - h\tilde{r} \end{pmatrix} = \tilde{V}^{[1]}.$$

The proof is completed. □

Combining propositions 1 and 2, we know that transformation (25) transforms U, $V^{[1]}$ to $\tilde{U}$, $\tilde{V}^{[1]}$ with the same form, respectively. The following theorem holds.

Theorem 1. *Let*

$$\tilde{r} = \frac{r-w}{1-v}, \qquad \tilde{s} = hw^{(1)} + x^{(1)} + s - x - \frac{r-w}{1-v}hv. \tag{38}$$

If r, s solve (2), then so do $\tilde{r}, \tilde{s}$ and transformation $(\phi; r, s) \to (\tilde{\phi}; \tilde{r}, \tilde{s})$ is a Darboux transformation of (2).

4. An exact solution of the RTL equation

Now let us apply the Darboux transformation to construct an exact solution of the RTL equation. Choose a seed solution $r = 1, s = 0$ of the RTL equation (2). Then the spectral problems (3) and (12) read

$$\phi^{(1)} = \begin{pmatrix} 0 & 1 \\ h\lambda - 1 & \lambda \end{pmatrix}\phi, \qquad \phi_t = \begin{pmatrix} -\frac{1}{2}\lambda - h & 1 \\ h\lambda - 1 & \frac{1}{2}\lambda - h \end{pmatrix}\phi, \tag{39}$$

which have two real basic solutions

$$\phi = \mathrm{e}^{(-h+\zeta)t}\begin{pmatrix} \left(\frac{1}{2}\lambda + \zeta\right)^n \\ \left(\frac{1}{2}\lambda + \zeta\right)^{n+1} \end{pmatrix}, \qquad \psi = \mathrm{e}^{-(h+\zeta)t}\begin{pmatrix} \left(\frac{1}{2}\lambda - \zeta\right)^n \\ \left(\frac{1}{2}\lambda - \zeta\right)^{n+1} \end{pmatrix},$$

where

$$\zeta = \sqrt{h\lambda - 1 + \tfrac{1}{4}\lambda^2}.$$

Thus we get

$$\begin{cases} \chi_1 = \dfrac{\left(\frac{1}{2}\lambda_1 + \zeta_1\right)^{n+1}\mathrm{e}^{2\zeta_1 t} - \gamma_1\left(\frac{1}{2}\lambda_1 - \zeta_1\right)^{n+1}}{\left(\frac{1}{2}\lambda_1 + \zeta_1\right)^{n}\mathrm{e}^{2\zeta_1 t} - \gamma_1\left(\frac{1}{2}\lambda_1 - \zeta_1\right)^{n}}, \\[2ex] \chi_2 = \dfrac{\left(\frac{1}{2}\lambda_2 + \zeta_2\right)^{n+1}\mathrm{e}^{2\zeta_2 t} - \gamma_2\left(\frac{1}{2}\lambda_2 - \zeta_2\right)^{n+1}}{\left(\frac{1}{2}\lambda_2 + \zeta_2\right)^{n}\mathrm{e}^{2\zeta_2 t} - \gamma_2\left(\frac{1}{2}\lambda_2 - \zeta_2\right)^{n}}, \end{cases} \tag{40}$$

where λ_1, λ_2 are arbitrary nonzero constants and

$$\zeta_i = \sqrt{h\lambda_i - 1 + \tfrac{1}{4}\lambda_i^2}, \qquad i = 1, 2.$$

Finally, we obtain a solution of the RTL equation (2)

$$\tilde{r} = 1 + \frac{\lambda_2 - \lambda_1}{\chi_1 - \chi_2} - \frac{1}{\rho_2}\chi_1\chi_2(\chi_1 - \chi_2 + \lambda_2 - \lambda_1)(\lambda_2 - \lambda_1),$$

$$\tilde{s} = \frac{\lambda_1 - \lambda_2}{\chi_1 - \chi_2} + \frac{\delta_1}{\delta_2} - \frac{\rho_1}{\rho_2},$$

where

$$\delta_1 = \lambda_2(h\lambda_2 - 1 + \lambda_2\chi_2)(h^2\lambda_1 - h - \chi_1) - \lambda_1(h\lambda_1 - 1 + \lambda_1\chi_1)(h^2\lambda_2 - h - \chi_2),$$

$$\delta_2 = (\chi_1 - \chi_2)(h\lambda_1 - 1)(h\lambda_2 - 1) + \chi_1\chi_2(\lambda_1 - \lambda_2),$$

$$\rho_1 = h(\lambda_1\chi_2 - \lambda_2\chi_1)(\lambda_2\chi_2 - \lambda_1\chi_1),$$

$$\rho_2 = h(\lambda_1\chi_2 - \lambda_2\chi_1)(\chi_1 - \chi_2) + (\chi_1 - \chi_2)^2.$$

5. Concluding remarks

In this paper, we have established a Darboux transformation for the RTL equation and obtained an explicit solution of the RTL equation. It is a pity that we have not known the interest in physics of the solution.

It is worth mentioning that starting from the spectral problem

$$E\phi = \tilde{U}\phi, \qquad \tilde{U} = \begin{pmatrix} 0 & 1 \\ (\alpha\lambda + \beta)r & \lambda + s \end{pmatrix}, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{41}$$

Ma and Xu [12] have derived the hierarchy of the combined TL–RTL equations. In an exactly analogous way, we can construct the Darboux transformation and explicit solutions of the hierarchy of the combined TL–RTL equations. We will report this result elsewhere.

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